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Knot polynomials from q-deformed algebras

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Abstract. We present explicit polynomials of two-knot invariants obtained from q-deformed algebras. Braid-group representations can be obtained from the R-matrices which in turn arise in q-deformed algebras. A Markov trace can be defined for R-matrices based on representations of the q-deformed algebras $su(n)_q$ and hence knot polynomials can be defined. In this paper, the properties of coupling coefficients and R-matrices based on each of the $\{1\}$ and $\{2\}$ representations for $su(n)_q$ are used to calculate polynomials for knots of ten or fewer crossings. We develop a new method to calculate the $\{2\}su(n)_q$ polynomials.

For the $\{1\}$ representation of $su(n)_q$, there are five pairs of knots of ten or fewer crossings which have the same polynomial. The exception is where $n = 2$. In this case the polynomial is equivalent to the Jones polynomial and has 14 pairs for knots of ten or fewer crossings. The $\{2\}su(n)_q$ polynomial has four pairs for these knots, each pair is different to the $\{1\}su(n)_q$ pairs. Thus, the $\{2\}su(n)_q$ polynomial has slightly fewer pairs than the $\{1\}su(n)_q$ polynomial and is significantly better at predicting the amphichirality or non-amphichirality of knots.

1. Introduction

The 1980s saw new developments in both knot theory and statistical mechanics. Given two knots or links, it is in general hard to tell whether they are topologically equivalent or not. Alexander [1] associated with each knot a polynomial invariant which distinguished some but not all. It was not until 1985 that Vaughan Jones discovered an improved polynomial [2]. This polynomial also indicated amphichirality or non-amphichirality, i.e. whether a knot is equivalent to its mirror image or not. However, the Jones polynomial does not distinguish all knots nor are all knots with symmetric Jones polynomials amphichiral. Since 1985, further improvements have been found by extending the Jones polynomial [3, 4]. These polynomials are mostly obtained by using braids to represent knots or links. In section 2, we outline the properties of knots and their braid representations.

The search for solutions to the Yang–Baxter equation in statistical mechanics led to the development of q-deformations of Lie algebras and other algebras [5–7]. Yang–Baxter equations without a spectral parameter have solutions, R-matrices, which may be found from representations of q-deformed algebras [8–10].

These topics in mathematics and physics were brought together with the realization that the multiplication rule of braid groups was equivalent to the Yang–Baxter equation without a spectral parameter [11]. New representations of braid groups were obtained from R-matrices. Markov traces can be defined on such representations leading to new knot and link polynomials [11–13]. The Markov trace and knot polynomial for braid representations obtained from R-matrices of the q-deformed algebras $su(n)_q$ are given in section 3.

Skein relations for invariants based on $su(n)_q$ representations are known [14] but very few explicit calculations of polynomials have been done. In this paper we calculate explicit

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polynomials based on the {1} and {2} representations of $su(n)_q$ for knots of ten or fewer crossings. We discuss the properties of these polynomials.

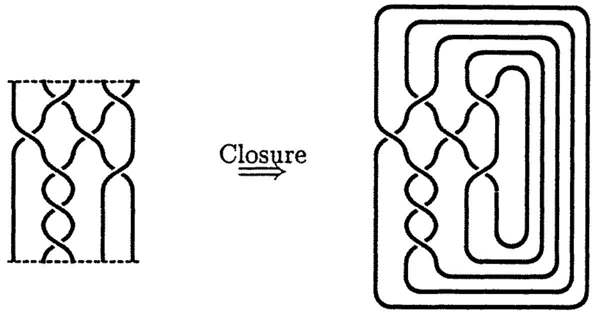


Figure 1. Closure of a braid to form a knot link.

The polynomial obtained from the fundamental representation {1} of $su(2)_q$ is equivalent to the Jones polynomial. For any n, the polynomial based on {1} of $su(n)_q$ is a particular case of the two-variable HOMFLY polynomial, an extension of the Jones polynomial. In this paper, we calculate explicit $\{1\}su(n)_q$ knot polynomials using the Alexander–Conway skein relation as a recursion relation. The properties of these polynomials are discussed in section 5.

For representations other than the fundamental, the skein relation is insufficient to calculate $su(n)_q$ polynomials. In section 4, we introduce a method for finding new skein-type relations. The properties of coupling coefficients and R-matrices are used to simplify the matrix representation of a braid. These relations, together with the Alexander–Conway skein relation, are used to calculate $\{2\}su(n)_q$ polynomials as described in section 6. The properties of the polynomials are discussed. The $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials for selected knots are given.

2. Knots and braids

Knots are smooth non-selfintersecting curves in $\mathbb{S}^3$. A knot invariant associates with each knot a more or less unique polynomial. One approach to find knot polynomials is through braids. An m-braid is a set of m strings between two parallel sets of m points arranged horizontally, as illustrated on the left-hand side of figure 1. The set of m-braids form a group, B_m, with concatenation as the group operation. An m-braid is generated by the set of single twists b_i and b_i^{-1} for $1 \leqslant i \leqslant m-1$ as shown in figures 2(a) and 2(b).

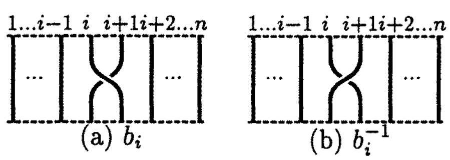


Figure 2. Braids and their generators.

An equivalent description of the braid group B_m [15] is that its generators satisfy the following relations

$$b_i b_j = b_j b_i \qquad |i-j| \geqslant 2 \tag{1}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{2}$$

Given a braid, its closure is found by connecting the corresponding points at top and bottom, as shown in figure 1. It is clear that the closure of a braid gives a knot or link. The description of knots by closed braids is highly non-unique. However, Markov [16] establishes that if two braids A' and A'' are obtained from the braid $A \in B_m$ by the moves given below, then the knots formed by the closure of the braids A, A', and A'' are topologically equivalent

$$\text{Markov I} \quad A' = BAB^{-1} \quad \text{for an arbitrary braid } B \in B_m \tag{3}$$

$$\text{Markov II} \quad A'' = Ab_m^{\pm 1} \quad \text{for } b_m, b_m^{-1} \in B_{m+1}. \tag{4}$$

These properties enable a knot polynomial to be found from the braid-group description of a knot. A knot polynomial $\alpha(A)$ obtained from some braid-group description, A, satisfying $\alpha(A) = \alpha(A') = \alpha(A'')$, for A', A'' as given in (3) and (4) is independent of the braid-group description used, depending only on the knot.

If a Markov trace ϕ on matrix representations $\mathbf{A}, \mathbf{B}$ of braids A, B can be defined so that it satisfies

$$\phi(\mathbf{A}) = \phi(\mathbf{BAB}^{-1}) \tag{5}$$

$$\phi(\mathbf{Ab}_m) = \tau\phi(\mathbf{A}) \qquad \phi(\mathbf{Ab}_m^{-1}) = \overline{\tau}\phi(\mathbf{A}) \qquad \text{for } A \in B_m,\ b_m, b_m^{-1} \in B_{m+1} \tag{6}$$

for τ and $\overline{\tau}$ such that $\tau = \phi(\mathbf{b}_i)$ and $\overline{\tau} = \phi(\mathbf{b}_i^{-1})$ for all i, then a knot polynomial can easily be obtained. The knot polynomial for the knot formed by the closure of the braid A is then

$$\alpha(\mathbf{A}) = (\tau\overline{\tau})^{-(m-1)/2} \left(\frac{\overline{\tau}}{\tau}\right)^{e(A)/2} \phi(\mathbf{A}) \tag{7}$$

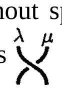
where $A \in B_m$ and $e(A)$ is the sum of the exponents of the generators b_i in A [11]. It is easily shown to be invariant under the Markov moves, equations (3) and (4).

3. R-matrices and $su(n)_q$ knot polynomials

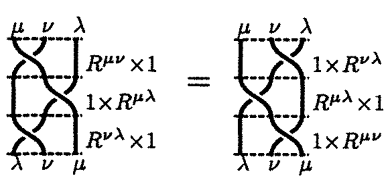
The q-deformed algebras $su(n)_q$ have a similar representation structure to the non-deformed algebras. Vector coupling coefficients can be defined for $su(n)_q$ in a similar way to $su(n)$ but these coefficients are dependent on the parameter q. In the limit as $q \to 1$, the $su(n)_q$ algebra reduces to $su(n)$. R-matrices describe the symmetry of the coupling coefficients when the coupling representations are interchanged [13].

$$(R^{\mu\nu})^{m_1 m_2}_{m_1' m_2'} \langle \mu m_1 \nu m_2 | r\lambda m\rangle = \{\mu\nu\lambda^* r\} q^{\{c(\mu)+c(\nu)-c(\lambda)\}/2} \langle \nu m_2' \mu m_1' | r\lambda m\rangle \tag{8}$$

where $c(\lambda)$ is the quadratic Casimir operator acting on λ and $\{\mu\nu\lambda^* r\}$ is a phase.

The R-matrices satisfy the Yang–Baxter equation without spectral parameter [13, 17]. We can represent the R-matrix $R^{\lambda\mu}$ in diagrammatic form as 

[13, 17]. The Yang–Baxter equation can be given diagrammatically as



(9)

When $\lambda = \mu = \nu$, Akutsu and Wadati [11] recognized that the Yang–Baxter equation is equivalent to the braid-group equation (2). A matrix representation, $\mathbf{b}_i$, for a generator b_i of the braid group B_n can thus be defined by

$$\mathbf{b}_i = \underbrace{1 \times \cdots \times \overbrace{R^{\lambda\lambda}}^{i\ \ i+1} \times \cdots \times 1}_{n} \tag{10}$$

with (1) being satisfied trivially and (2) following from the Yang–Baxter equation.

A Markov trace can be obtained for the braid-group representation obtained from a representation λ of the q-deformed algebra $su(n)_q$. We first define an enhancement matrix $\mathbf{V}$ by

$$\mathbf{V} = \underbrace{v_\lambda \times \cdots \times v_\lambda}_{n} \qquad v_\lambda = \mathrm{diag}\{q^{\rho(\Lambda)} | \Lambda \text{ a weight of } \lambda\} \tag{11}$$

where $\rho = \frac{1}{2}\sum_\alpha H_\alpha$, α a positive root of $su(n)$, and the braid group under consideration has dimension n.

The R-matrix $R^{\lambda\lambda}$ and the matrix v_λ satisfy the following relations

$$R^{\lambda\lambda}(v_\lambda \times v_\lambda) = (v_\lambda \times v_\lambda)R^{\lambda\lambda} \tag{12}$$

$$\mathrm{tr}_2((1 \times v_\lambda)R^{\lambda\lambda}) = q^{-c(\lambda)}1 \tag{13}$$

$$\mathrm{tr}(v_\lambda) = |\lambda|. \tag{14}$$

Equation (14) defines the q-dimension $|\lambda|$ of the representation λ. The trace in (13) is over the second space. Proofs for (12) and (13) are given in Reshetikhin [13] and Zhang *et al* [18].

The modified trace defined below is a Markov trace

$$\phi(A) = \frac{1}{|\lambda|^m}\mathrm{tr}(\mathbf{VA}) \qquad \tau = \frac{q^{-c(\lambda)}}{|\lambda|} \qquad \overline{\tau} = \frac{q^{c(\lambda)}}{|\lambda|} \tag{15}$$

with (5) following from (12) and (6) following from (13). The knot polynomial for the braid $A \in B_n$ from (7) and (15) is thus

$$\alpha(A) = |\lambda|^{-1} q^{e(A)c(\lambda)}\mathrm{tr}(\mathbf{VA}). \tag{16}$$

The polynomial is normalized so that for the unknot (trivial knot or circle) $\alpha(b_i) = \alpha(1) = \alpha(b_i^{-1}) = 1$ where 1 is the trivial (and only) braid in B_1.

From (8), the R-matrices $R^{\lambda\lambda}$ have eigenvalues $q^{2c(\lambda)-c(\mu)/2}$, where $\mu \subset \lambda \times \lambda$. Thus $R_q^{\lambda\lambda}$ satisfies

$$\prod_{\mu \subset \lambda \times \lambda} ((R^{\lambda\lambda})^r - \{\lambda\lambda\mu^*\}^r q^{r(2c(\lambda)-c(\mu))/2}) = 0 \tag{17}$$

where r is the multiplicity of μ in $\lambda \times \lambda$. This leads to a relation of the form below for knot polynomials based on R-matrices for arbitrary braids A, B

$$\alpha(\mathbf{Ab}_i^k\mathbf{B}) = h_{k-1}\alpha(\mathbf{Ab}_i^{k-1}\mathbf{B}) + \cdots + h_1\alpha(\mathbf{Ab}_i\mathbf{B}) + h_0\alpha(\mathbf{AB}). \tag{18}$$

Such relations are called Alexander–Conway skein relations. Skein relations are used in the following sections as recursion relations.

4. Skein-type relations

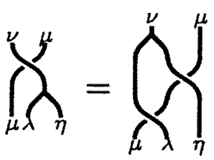
In principle, polynomials for knots based on any representation of $su(n)_q$ can be obtained from (16). However, for all but the simplest knots, the matrix multiplications are long and tedious. The skein relation is sufficient to calculate all polynomials for the fundamental representation of $su(n)_q$ by recursion. For other representations the skein relation is not sufficient for complete calculation but can be used in simplification.

Guadagnini [19] outlines a new method for calculating $su(n)_q$ knot polynomials. The knot polynomial for a particular representation is reduced to a sum over polynomials for trivial knots based on other representations. The properties of Wilson line operators are used in the simplification process.

In this section, we use a similar approach to Guadagnini, but with two major differences. First, the properties of R-matrices and vector coupling coefficients are used. These arise naturally in the deformed algebra. Secondly, rather than tackling each knot individually, we derive two 'skein-type' relations which can be used to calculate polynomials for whole classes of braids.

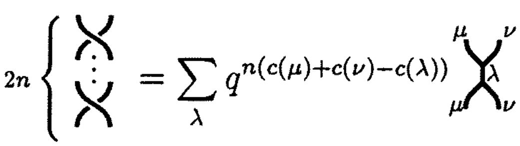
In addition to the Yang–Baxter equation, there are further relations for the R-matrices and coupling coefficients as summarized below, where in a similar manner to the R-matrices the coupling coefficient $\langle \mu\nu|\lambda\rangle$ and its conjugate $\langle\lambda|\mu\nu\rangle$ are given diagrammatically as 
and 

$$(R^{\nu\mu})^{m_3 m}_{m_3' m''}\langle\lambda m_1'\eta m_2'|\nu m_3'\rangle = \langle\lambda m_1\eta m_2|\nu m_3\rangle (R^{\eta\mu})^{m_2 m}_{m_2' m'}(R^{\lambda\mu})^{m_1 m'}_{m_1' m''} \tag{19}$$



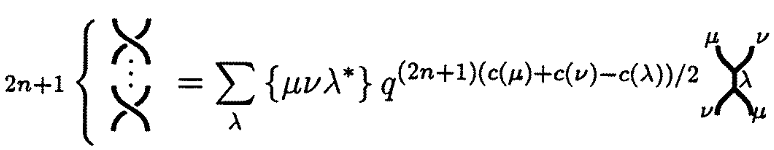
(19′)

$$\underbrace{(R^{\mu\nu})^{m_1m_2}_{k_1k_2}(R^{\nu\mu})^{k_1k_2}_{k_2'k_1'}\dots(R^{\nu\mu})^{k_2''k_1''}_{m_2'm_1'}}_{2n} = \sum_{\lambda\in\mu\times\nu} q^{n(c(\mu)+c(\nu)-c(\lambda))}\langle\mu m_1\nu m_2|\lambda m\rangle\langle\lambda m|\mu m_1'\nu m_2'\rangle \tag{20}$$



(20′)

$$\underbrace{(R^{\mu\nu})^{m_1m_2}_{k_1k_2}\dots(R^{\mu\nu})^{k_1''k_2''}_{m_1'm_2'}}_{2n+1} = \sum_{\lambda\in\mu\times\nu}\left\{\mu\nu\lambda^*\right\} q^{(2n+1)(c(\mu)+c(\nu)-c(\lambda))/2}\langle\mu m_1\nu m_2|\lambda m\rangle\langle\lambda m|\nu m_1'\mu m_2'\rangle \tag{21}$$



(21′)

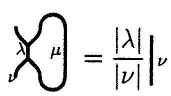
Equation (19) is the pentagonal equation [13, 17, 20]. Equations (20) and (21) follow from the definition of the R-matrix, (8), and the two vector coupling coefficient orthogonality relations.

In order to use the above relations in the simplification process, the idea of a braid is extended. The coloured-braid group associates 'colours' with each string. In this case, the strings are associated with various irreps of $su(n)_q$. The crossing of two strings associated with irreps λ and μ is generated with the R-matrix $R^{\lambda\mu}$. The general R-matrices generate representations of the coloured-braid group. When all the strings are associated with the same irrep the previous case is retrieved.

In order for closure to remain sensible, each pair of strings being joined must be associated with the same representation. For a braid to represent a knot, therefore, all strings must be associated with a single representation.

For braids with sensible closure, the Markov trace ϕ generalizes. If the strings are associated with representations μ, ν etc then $\phi_{\mu\nu\ldots} = (|\mu||\nu|\ldots)^{-1}\mathrm{tr}(\mathbf{V}_{\mu\nu\ldots}\mathbf{A})$ where $\mathbf{V}_{\mu\nu\ldots} = v_\mu \times v_\nu \times \cdots$. From the definition of the Markov trace, taking the closure of a braid is equivalent to taking a modified trace of the product of R-matrices representing the braid. The modified trace can be performed on other matrices, extending the idea of closure. When it is applied to coupling coefficients, the following relation holds

$$\mathrm{tr}(V_\mu\langle\mu\nu|\lambda\rangle\langle\lambda|\mu\nu\rangle) = \frac{|\lambda|}{|\nu|} \tag{22}$$



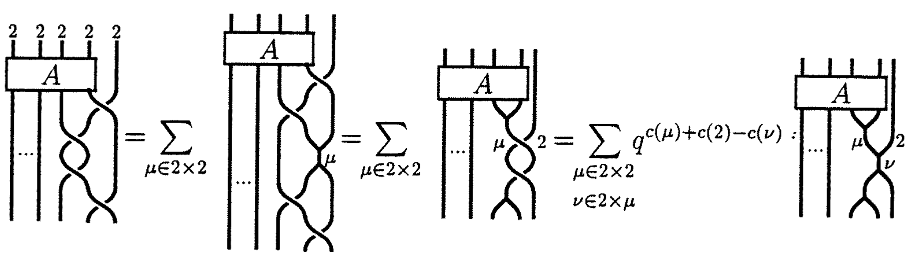
(22′)

Equation (22) follows from the definition of the enhancement matrix v and the orthogonality and symmetries of the coupling coefficients [9].

The two skein-type relations used in the calculation of $\{2\}su(n)_q$ polynomials reduce a braid of dimension $(m+1)$ or $(m+2)$ to a sum over braids of dimension m. One or two strings may be eliminated by reducing crossings with (20) and (21), manipulating with the pentagonal equation (19) and Yang–Baxter equation (9) and finally using the closure equivalence.

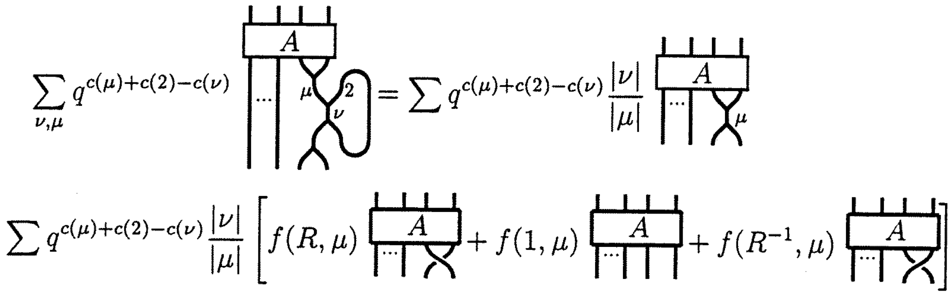
To illustrate this method, the calculation of the skein-type relation, (28), is outlined below

$\mathsf{A}\mathsf{b}_n\mathsf{b}_{n-1}^2\mathsf{b}_n =$



(23)

The first step uses coupling coefficient orthogonality. The pentagonal equation, (19), is then repeatedly applied. The final step uses (20) to write the product of two R-matrices as a second sum over coupling coefficients. On closure and using (22) we have



(24)

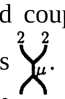
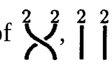
where in the final step the definitions of R^{22} and $(R^{22})^{-1}$, (8), and coupling coefficient orthogonality, are used to give three expressions for the three products . These equations are rearranged to give the coefficients f for writing the product  in terms of  and . Knot polynomials can now be obtained as $\{2\}$ is the only representation involved. The polynomial $\alpha(Ab_m b_{m+1} b_m)$ can thus be written as a sum over the polynomials $\alpha(Ab_m)$, $\alpha(A)$ and $\alpha(Ab_{m-1})$. The coefficients are given in table 1. A similar approach is used for the second skein-type relation, equation (29), with two strings being eliminated.

Table 1. Coefficients for skein-type relations, equations (28) and (29).

h_1	$-q^{3n+2} + q^{2n+4} - 2q^{2n+3} + q^{2n+2} + 2q^{2n+1} - 2q^{2n} + q^{2n-2}$
h_0	$q^{5n+6} - q^{4n+8} - q^{4n+5} + q^{4n+4} - q^{4n+2} + 2q^{3n+8} - 3q^{3n+7} - 2q^{3n+6} + 10q^{3n+5} - 6q^{3n+4} - 7q^{3n+3}$ $+10q^{3n+2} - q^{3n+1} - 4q^{3n} + 2q^{3n-1})/(q-1)^2(q+1)$
h_{-1}	$q^{5n+4} - 2q^{4n+4} + 2q^{4n+3} + q^{4n+2} - 2q^{4n+1}$
k_1	$2q^{10n+10} + (-q^{12} - q^{11} - 6q^{10} + 4q^9 + 5q^8 - 5q^7 - 2q^6)q^{9n} + (4q^{12} - 6q^{10} + 6q^9 - 4q^7 + 8q^6$ $-6q^4 + 2q^3 + 2q^2)q^{8n} + (-q^{13} + q^{12} - q^{11} + q^{10} - q^9 + 2q^7 - 5q^6 + q^5 + 4q^4 - 2q^3 - q^2 - q + 1$ $+q^{-1} - q^{-2})q^{7n}/(q-1)^2(q+1)$
k_0	$q^{10n+12} + (-2q^{14} - 2q^{11} + 2q^{10} - 2q^8)q^{9n} + (q^{16} + 5q^{14} - 6q^{13} - 9q^{12} + 28q^{11} - 7q^{10} - 28q^9 + 23q^8$ $+8q^7 - 13q^6 + 2q^5 + 2q^4)q^{8n} + (-4q^{16} + 4q^{15} + 10q^{14} - 22q^{13} - 2q^{12} + 30q^{11} - 20q^{10}$ $-6q^9 + 20q^8 - 24q^7 + 26q^5 - 12q^4 - 12q^3 + 8q^2 + 2q - 2)q^{7n} + (q^{17} - 2q^{16} + 4q^{14} - q^{13} - 5q^{12}$ $-2q^{11} + 17q^{10} - 10q^9 - 19q^8 + 30q^7 - 5q^6 - 20q^5 + 16q^4 - 2q^3 + q - 7 + 6q^{-1} + q^{-2} - 3q^{-3} + q^{-4})$ $\times q^{6n}/(q-1)^4(q+1)^2$
k_{-1}	$-2q^{8n+8} + (4q^{10} - 4q^9 + q^8 + 7q^7 - 4q^6 - 2q^5 + 3q^4 + q^3)q^{7n} + (-2q^{12} + 2q^{11} + 2q^{10} - 6q^9$ $+4q^5 - 2q^4 - 6q^3 + 4q - 2q^{-1})q^{6n} + (q^{13} - 2q^{12} + 3q^{10} + q^9 - 5q^8 + q^7 + 5q^6 - 5q^5 + 2q^4 + 4q^3$ $-5q^2 + q + 3q^{-2} - 2q^{-3} - q^{-4} + q^{-5})q^{5n}/(q-1)^2(q+1)$

5. Knot polynomials based on the fundamental representation of $su(n)_q$

The fundamental representation of $su(n)_q$ can be written in partition form as $\{1, 0, \ldots, 0\}$ and is m-dimensional. In the following sections, it is written as $\{1\}su(n)_q$. The tensor product of the fundamental representation with itself decomposes into two terms, the symmetric term $\{2\}$ and the antisymmetric term $\{1\,1\}$. Using (7), (15), and (17), one obtains a two-term skein relation for the $\{1\}su(n)_q$ polynomials as

$$\alpha(Ab_i^2 B) = (q^{(n-1)/2} - q^{(n+1)/2})\alpha(Ab_i B) + q^n \alpha(AB). \tag{25}$$

To calculate the $\{1\}su(n)_q$ polynomial for a knot, the braid representing the knot is written as a product of p terms of form $b_1^{k_1}b_2^{k_2}\ldots b_n^{k_m}$, that is as $b_1^{l_{11}}b_2^{l_{12}}\ldots b_m^{l_{1m}}\ldots b_1^{l_{p1}}b_2^{l_{p2}}\ldots b_n^{l_{pm}}$. If the last non-zero power is l_{pi}, that is $l_{pi+1} = \cdots = l_{pm} = 0$, the polynomial can be written as a sum of polynomials with last non-zero power in the $i-1$ position or lower by means of the skein relation, braid-group relations and Markov moves. When $i = 1$, by the Markov move, (3), the $b_1^{l_{p1}}$ term can be absorbed with the $b_1^{l_{11}}$ term and hence the number of terms p is reduced by 1. The recursion continues until $p = 1$ when $\alpha(b_1^{k_1}\ldots b_m^{k_m}) = P_{k_1}P_{k_2}\ldots P_{k_m}$, where $P_k = \alpha(b^k)$. For each k, P_k is found from iterating the skein relation with A, B the trivial braids and noting that $P_1 = \alpha(b_1) = P_{-1} = \alpha(b_1^{-1}) = 1$.

The knots were sorted according to braid index, i.e. the dimension of the braid representation of the knot, and number of terms p. The braid words were taken from [21]. An algebraic package, MAPLE [22], was then used to carry out the above steps.

For $n = 1$ the polynomials for all knots are equal to 1 and are thus identical. For higher values of n, all polynomials are different except for the occasional pair. For $n = 2$, among the 248 knots of ten or fewer crossings, there are 14 pairs of knots having the same polynomials. Of these 14 pairs, five are pairs of knots which have the same polynomial for all values of n. For $n > 2$, all knots of ten or fewer crossings were distinguished by the $\{1\}su(n)_q$ polynomial with the exception of these five pairs.

The $\{1\}su(n)_q$ polynomials are a special case of the HOMFLY polynomial [3]. This two-variable polynomial has the skein relation $t^{-1}P(Ab_iB) - tP(Ab_i^{-1}B) = xP(AB)$. With $t = q^{n/2}$ and $x = q^{-1/2} - q^{1/2}$, the skein relation of the $\{1\}su(n)_q$ polynomials is obtained. The pairs of knots of ten or fewer crossings which cannot be distinguished by the $\{1\}su(n)_q$ polynomial are exactly those with the same HOMFLY polynomial. For knots of ten or fewer crossings, the one variable $\{1\}su(n)_q$ polynomials for any $n > 2$ distinguish the same knots as the two variable HOMFLY polynomial.

For special values of the deformation parameter q, the $\{1\}su(n)_q$ polynomials take on certain discrete values. If $q^{(n-1)/2} = 1$ or $q^{(n+1)/2} = -1$ then $\alpha(A) = 1$ for all braids A, as is readily shown from the skein relation. Likewise, using induction on the skein relation, we can show that for $q^{(n-1)/2} = -1$ or $q^{(n+1)/2} = -1$, the polynomial $\alpha(A)$ equals 1 for braids A that describe a knot or an odd-component link, $\alpha(A)$ equals -1 for the even-component case.

These special values mean that for any knot with polynomial $\alpha(A)$, the polynomial $1 - \alpha(A)$ has factors $(q^{(n+1)/2} - q^{-(n+1)/2})$ and $(q^{(n-1)/2} - q^{-(n-1)/2})$. This is used to simplify the knot polynomials. The reduced knot polynomials

$$\frac{1-\alpha(A)}{(q^{(n+1)/2} - q^{-(n+1)/2})(q^{(n-1)/2} - q^{-(n-1)/2})}$$

for selected knots are given in table 3. Setting $n = 2$, we recover the table given by Jones [21] (on replacing q for t).

6. Knot polynomials based on $\{2\}$ of $su(n)_q$

Knot polynomials based on the non-fundamental representations of $su(n)_q$ have skein relations of power, N, greater than 2. The Alexander–Conway skein relation is insufficient to determine all knot polynomials. Either the braid representations must be entered explictly into the definition of the knot polynomial, (16), or the skein relation must be supplemented with other skein-type relations. All of the $\{2\}su(n)_q$ polynomials for knots of ten or fewer crossings can be calculated using the Alexander–Conway skein relation together with two other skein-type relations.

The $\{2\}su(n)_q$ polynomials are of power $N = 3$, as there are three terms in the product

$$\{2\} \times \{2\} = \{4\} + \{31\} + \{22\}. \tag{26}$$

From (7), (15) and (17), the $\{2\}su(n)_q$ skein relation is

$$\alpha(Ab_i^2B) = q^n(q^2 - q + q^{-1})\alpha(Ab_iB) + q^{2n}(q^3 - q + 1)\alpha(AB) - q^{3n+2}\alpha(Ab_i^{-1}B). \tag{27}$$

Table 2. Summary of pairs and falsely amphichiral knots.

Invariant	Knots which are not distinguished
$\{1\}su(n)_q$	5_1–10_{132}, 8_8–10_{129}, 8_{16}–10_{156}, 10_{25}–10_{56}, 10_{40}–10_{103}
$\{1\}su(2)_q$(additional)	10_{22}–10_{35}, 10_{41}–10_{94}, 10_{43}–10_{91}, 10_{59}–10_{106}, 10_{60}–10_{83}, 10_{71}–10_{104}, 10_{73}–10_{86}, 10_{81}–10_{109}, 10_{137}–10_{155}
$\{2\}su(n)_q$	3_1–7_7, 7_6–10_{60}, 8_{11}–10_7, 9_{44}–10_{71}
Invariant	**Falsely amphichiral knots**
$\{1\}su(n)_q$	9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}, 10_{125}
$\{2\}su(n)_q$	None

The two relations below are needed in addition to calculate $\{2\}su(n)_q$ polynomials for all knot of ten or fewer crossings where the coefficients $k_1, \dots, h_{-1}$ are given in table 1.

$$\alpha(Ab_nb_{m-1}^2b_m) = h_1\alpha(Ab_{m-1}) + h_0\alpha(A) + h_{-1}\alpha(Ab_{m-1}^{-1}) \qquad \text{where } A \in B_m \tag{28}$$

$$\alpha(Ab_mb_{m-1}b_{m+1}b_m^2b_{m-1}b_{m+1}b_m) = k_1\alpha(Ab_{m-1}) + k_0\alpha(A) + k_{-1}\alpha(Ab_{m-1}^{-1}) \quad \text{where } A \in B_m. \tag{29}$$

The method used in obtaining these relations is outlined in section 4 with (28) used as an illustration.

The $\{2\}su(n)_q$ knot polynomials are calculated in a similar manner to the $\{1\}su(n)_q$ polynomials. The skein relation, (27), is used to reduce the knot polynomial to a sum over simpler knot polynomials with the two skein-type relations, (28) and (29), being used when no other simplification is possible.

The $\{2\}su(n)_q$ polynomial can be factored in a similar manner to the $\{1\}su(n)$ polynomials. For any braid A, it follows from the skein relation, (27), and the two skein-type relations, (28) and (29), that $\alpha(A) = 1$ for $q^{n+2} = 1$ or $q^{n-1} = 1$. Thus any knot with polynomial $\alpha(A)$ has factors $(q^{(n+2)/2} - q^{-(n+2)/2})(q^{(n-1)/2} - q^{-(n-1)/2})$ for $1 - \alpha(A)$. This provides a check on the calculation of the polynomials.

Hou *et al* [17] show that the $\{2\}su(2)_q$ polynomial is equivalent to the polynomial of Akutsu and Wadati [23] found from a three-state exactly solvable model in statistical mechanics. The knot polynomials calculated here for knots with braid index 2 or 3 were compared for $n = 2$ to those of Akutsu and Wadati [23]. The values given by Akutsu and Wadati for $\alpha(10_{100})$ and $\alpha(10_{112})$ have incorrect factors for $1 - \alpha(A)$. These two polynomials are among those given in table 3.

The $\{2\}su(n)_q$ polynomial distinguishes all of the pairs of knots of the $\{1\}su(n)_q$ polynomials for knots of ten or fewer crossings, both the five pairs for all n and the nine further pairs only for $n = 2$. However, the $\{2\}su(n)_q$ polynomial has four pairs for knots of ten or fewer crossings. They are: 3_1 and 7_7, 7_6 and 10_{60}, 8_{11} and 10_7, 9_{44} and 10_{71}. These polynomials are given in table 3.

Table 3. $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials. The table gives for selected knots of ten or fewer crossings their braid index (BI), reduced $\{1\}su(n)_q$ polynomial and reduced $\{2\}su(n)_q$ polynomial, together with notes on these. The knots given are those mentioned elsewhere in the paper, that is, those which are part of a pair or have polynomials of interest. The reduced $\{1\}su(n)_q$ polynomials given are $(1-\alpha(A))/(q^{(n+1)/2}-q^{-(n+1)/2})(q^{(n-1)/2}-q^{-(n-1)/2})$, where $\alpha(A)$ is the $\{1\}su(n)_q$ polynomial for the knot A. The terms in brackets are coefficients of $1, q^{\pm 1}, q^{\pm 2}, \ldots$, the overlined terms being negative coefficients. For example, the reduced $\{1\}su(n)_q$ polynomial for 7_7 is $-q^n - 2 + q + q^{-1}$. In a similar manner, the reduced $\{2\}su(n)_q$ polynomials given are $(1-\alpha(A))/(q^{(n+2)/2}-q^{-(n+2)/2})(q^{(n-1)/2}-q^{-(n-1)/2})$. The terms in brackets are coefficients of $1, q, q^2 \ldots$ (notice the difference to the reduced $\{1\}su(n)_q$ polynomials). The reduced $\{2\}su(n)_q$ polynomial for 7_7 is thus $q^{n+\frac{1}{2}} + q^{2n-\frac{1}{2}} + q^{2n+\frac{5}{2}} - q^{3n+\frac{5}{2}}$.

Knot	BI	$\{1\}su(n)_q$ polynomial	Comments
		$\{2\}su(n)_q$ polynomial	Comments
3_1	2	$q^n(1)$	
		$q^{n+\frac{1}{2}}(1) + q^{2n-\frac{1}{2}}(1001) + q^{3n+\frac{5}{2}}(\bar{1})$	Same as 7_7
5_1	2	$q^n(1) + q^{2n}(01)$	Same as 10_{132}, all n
		$q^{n+\frac{1}{2}}(1) + q^{2n-\frac{1}{2}}(1001) + q^{3n-\frac{3}{2}}(1001001) + q^{4n-\frac{5}{2}}(1001001001) + q^{5n+\frac{1}{2}}(\bar{1}00\bar{1}\bar{1}0\bar{1})$	
7_6	4	$q^n(2\bar{1}) + q^{2n}(1)$	
		$q^{n-\frac{7}{2}}(\bar{1}11\bar{2}11\bar{1}) + q^{2n-\frac{7}{2}}(1\bar{2}\bar{1}6\bar{2}\bar{4}50\bar{2}1) + q^{3n-\frac{3}{2}}(\bar{1}\bar{1}40\bar{5}22\bar{2}) + q^{4n+\frac{3}{2}}(20\bar{2}11) + q^{5n+\frac{9}{2}}(\bar{1})$	Same as 10_{60}
7_7	4	$(\bar{2}1) + q^n(\bar{1})$	
		$q^{n+\frac{1}{2}}(1) + q^{2n-\frac{1}{2}}(1001) + q^{3n+\frac{5}{2}}(\bar{1})$	Same as 3_1
8_8	4	$(\bar{1}1) + q^n(\bar{1}1)$	Same as 10_{129}, all n
		$q^{-n-\frac{9}{2}}(\bar{1}11\bar{2}01\bar{1}) + q^{-\frac{7}{2}}(\bar{1}22\bar{4}04\bar{1}\bar{1}1) + q^{n-\frac{9}{2}}(10\bar{2}42\bar{6}25\bar{2}\bar{1}1) + q^{2n-\frac{5}{2}}(1\bar{2}04\bar{4}\bar{2}5\bar{1}\bar{2}1) + q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	
8_{11}	4	$q^n(2\bar{1}) + q^{2n}(1\bar{1})$	
		$q^{n-\frac{7}{2}}(\bar{1}11\bar{2}11\bar{1}) + q^{2n-\frac{5}{2}}(\bar{3}06\bar{4}\bar{4}5\bar{1}\bar{2}1) + q^{3n-\frac{7}{2}}(10\bar{4}17\bar{4}\bar{6}50\bar{2}1) + q^{4n-\frac{3}{2}}(1\bar{2}\bar{1}6\bar{1}\bar{5}41\bar{2}1) + q^{5n+\frac{3}{2}}(\bar{1}11\bar{2}01\bar{1})$	Same as 10_7
8_{16}	3	$q^n(3\bar{2}1)$	Same as 10_{156}, all n
		$q^{n-\frac{13}{2}}(\bar{1}22\bar{5}16\bar{5}\bar{1}5\bar{3}\bar{1}2\bar{1}) + q^{2n-\frac{13}{2}}(1\bar{2}\bar{1}7\bar{5}\bar{9}141\overline{14}103\bar{8}51\bar{2}1) + q^{3n-\frac{7}{2}}(\bar{1}21\bar{6}37\bar{9}\bar{1}7\bar{4}\bar{1}2\bar{1})$	
9_{42}	4	$(0\bar{1})$	Falsely amphichiral
		$q^{-n-\frac{11}{2}}(\bar{1}00\bar{1}\bar{1}0\bar{1}) + q^{-\frac{11}{2}}(1001100001) + q^{n-\frac{5}{2}}(\bar{1}\bar{1}01\bar{2}\bar{1}) + q^{2n+\frac{3}{2}}(10\bar{1})$	

Table 3. (Continued)

Knot	BI	$\{1\}su(n)_q$ polynomial $\{2\}su(n)_q$ polynomial	Comments Comments
9_{44}	4	$(\bar{1})+q^{n}(\bar{1}1)$	
		$q^{-n-\frac{3}{2}}(\bar{1})+q^{-\frac{7}{2}}(110\bar{1}01)+q^{n-\frac{7}{2}}(\bar{1}\bar{1}3\bar{2}\bar{4}21\bar{2})+q^{2n-\frac{5}{2}}(1\bar{1}13\bar{3}04\bar{1}\bar{1}1)+q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	Same as 10_{71}
10_{7}	5	$q^{n}(2\bar{1})+q^{2n}(2\bar{1})+q^{3n}(1\bar{1})$	
		$q^{n-\frac{7}{2}}(\bar{1}11\bar{2}111\bar{1})+q^{2n-\frac{5}{2}}(\bar{3}06\bar{4}\bar{4}5\bar{1}\bar{2}1)+q^{3n-\frac{7}{2}}(10\bar{4}17\bar{4}\bar{6}50\bar{2}1)+q^{4n-\frac{3}{2}}(1\bar{2}\bar{1}6\bar{1}\bar{5}41\bar{2}1)+q^{5n+\frac{3}{2}}(\bar{1}11\bar{2}01\bar{1})$	Same as 8_{11}
10_{22}	4	$(\bar{2}1\bar{1})+q^{n}(\bar{2}1\bar{1})$	Same as 10_{35}, $n=2$
		$q^{-n-\frac{15}{2}}(\bar{1}10\bar{3}12\bar{4}02\bar{2}01\bar{1})+q^{-\frac{13}{2}}(\bar{1}10\bar{4}33\bar{8}06\bar{4}\bar{2}3\bar{1}\bar{1}1)+q^{n-\frac{15}{2}}(10\bar{2}31\bar{6}74\overline{12}49\bar{7}\bar{2}5\bar{1}\bar{1}1)+q^{2n-\frac{11}{2}}(1\bar{2}04\bar{5}08\bar{8}\bar{3}10\bar{4}\bar{5}50\bar{2}1)$ $+q^{3n-\frac{5}{2}}(\bar{1}10\bar{2}20\bar{4}21\bar{3}01\bar{1})$	
10_{25}	4	$q^{n}(1)+q^{2n}(\bar{2}2\bar{1})+q^{3n}(\bar{3}2\bar{1})$	Same as 10_{56}, all n
		$q^{n+\frac{1}{2}}(1)+q^{2n-\frac{1}{2}}(1001)+q^{3n-\frac{11}{2}}(\bar{1}11\bar{3}13\bar{3}02\bar{2}01\bar{1})+q^{4n-\frac{9}{2}}(\bar{3}17\bar{8}\bar{6}13\bar{4}\overline{10}10\bar{1}\bar{7}50\bar{2}1)+q^{5n-\frac{11}{2}}(10\bar{5}312\overline{12}\overline{11}22\bar{2}\overline{18}143\overline{12}62\bar{3}1)$ $+q^{6n-\frac{7}{2}}(1\bar{3}\bar{1}11\bar{6}\overline{15}195\overline{21}118\overline{12}43\bar{3}1)+q^{7n-\frac{1}{2}}(\bar{1}21\bar{6}36\bar{8}06\bar{4}\bar{1}2\bar{1})$	
10_{35}	6	$q^{-n}(\bar{1})+(\bar{3}1)+q^{n}(\bar{3}1)+q^{2n}(\bar{1})$	Same as 10_{22}, $n=2$
		$q^{-3n-\frac{7}{2}}(\bar{1})+q^{-2n-\frac{9}{2}}(11\bar{3}\bar{1}2)+q^{-n-\frac{9}{2}}(\bar{1}32\bar{8}\bar{2}50\bar{1})+q^{-\frac{9}{2}}(\bar{1}\bar{1}63\overline{14}\bar{3}11\bar{1}\bar{3}1)+q^{n-\frac{9}{2}}(1\bar{2}\bar{3}113\overline{19}013\bar{2}\bar{3}1)+q^{2n-\frac{3}{2}}(\bar{3}48\overline{14}\bar{7}130\bar{4}1)$ $+q^{3n-\frac{1}{2}}(\bar{1}15\bar{4}\bar{7}53\bar{2})+q^{4n+\frac{5}{2}}(2\bar{1}\bar{3}11)+q^{5n+\frac{11}{2}}(\bar{1})$	
10_{40}	4	$q^{n}(4\bar{2}1)+q^{2n}(3\bar{2}1)$	Same as 10_{103}, all n
		$q^{n-\frac{13}{2}}(\bar{1}21\bar{6}36\bar{8}16\bar{4}\bar{1}2\bar{1})+q^{2n-\frac{11}{2}}(\bar{2}45\overline{15}224\overline{17}\overline{11}23\bar{5}\bar{9}80\bar{2}1)+q^{3n-\frac{13}{2}}(10\bar{4}68\overline{20}032\overline{19}\overline{21}28\bar{2}\overline{14}91\bar{3}1)+q^{4n-\frac{9}{2}}(1\bar{3}09\overline{11}\bar{9}25\bar{7}\overline{23}204\overline{14}62\bar{3}1)$ $+q^{5n-\frac{3}{2}}(\bar{1}20\bar{5}53\bar{9}35\bar{5}02\bar{1})$	
10_{41}	5	$(1\bar{1})+q^{n}(4\bar{3}1)+q^{2n}(1\bar{1})$	Same as 10_{94}, $n=2$
		$q^{-n-\frac{11}{2}}(\bar{1}11\bar{2}01\bar{1})+q^{-\frac{13}{2}}(11\bar{5}09\bar{5}\bar{6}7\bar{1}\bar{3}2)+q^{n-\frac{13}{2}}(\bar{2}35\overline{12}\bar{5}20\bar{4}\overline{17}113\bar{8}31\bar{1})+q^{2n-\frac{13}{2}}(1\bar{3}112\overline{13}\overline{16}304\overline{30}1410\overline{14}53\bar{3}1)$ $+q^{3n-\frac{9}{2}}(\bar{1}06\bar{6}\overline{13}179\overline{23}412\overline{10}04\bar{2})+q^{4n-\frac{3}{2}}(2\bar{2}\bar{4}73\bar{8}24\bar{3}01)+q^{5n+\frac{3}{2}}(\bar{1}11\bar{2}01\bar{1})$	

Table 3. (Continued)

Knot	BI	$\{1\}su(n)_q$ polynomial	Comments
		$\{2\}su(n)_q$ polynomial	Comments
10_{43}	5	$q^{n}(\bar{1}1) + (\bar{4}3\bar{1}) + q^{-n}(\bar{1}1)$	Same as 10_{91}, $n = 2$
		$q^{-3n-\frac{13}{2}}(\bar{1}11\bar{2}01\bar{1}) + q^{-2n-\frac{15}{2}}(10\bar{4}36\bar{7}\bar{2}7\bar{2}\bar{2}2) + q^{-n-\frac{15}{2}}(\bar{2}42\overline{14}718\overline{20}\bar{7}19\bar{5}\bar{7}50\bar{1}) + q^{-\frac{15}{2}}(1\bar{3}29\overline{17}\bar{1}33\overline{22}\overline{22}33\bar{1}\overline{17}92\bar{3}1)$ $+q^{n-\frac{11}{2}}(\bar{1}05\bar{7}\bar{5}19\bar{7}\overline{20}187\overline{14}24\bar{2}) + q^{2n-\frac{5}{2}}(2\bar{2}\bar{2}7\bar{2}\bar{7}63\bar{4}01) + q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	
10_{48}	3	$(\bar{2}3\bar{1}1)$	Falsely amphichiral
		$q^{-n-\frac{21}{2}}(\bar{1}1\bar{1}\bar{3}30\bar{8}34\bar{9}\bar{1}4\bar{6}01\bar{3}01\bar{1}) + q^{-\frac{21}{2}}(1\bar{1}14\bar{3}210\bar{6}019\bar{5}\bar{6}180\bar{5}90\bar{2}40\bar{1}1) + q^{n-\frac{15}{2}}(\bar{1}1\bar{1}\bar{3}2\bar{1}\bar{7}40\bar{9}33\bar{7}12\bar{3}01\bar{1})$	
10_{56}	4	$q^{n}(1) + q^{2n}(\bar{2}2\bar{1}) + q^{3n}(\bar{3}2\bar{1})$	Same as 10_{25}, all n
		$q^{n+\frac{1}{2}}(1) + q^{2n-\frac{1}{2}}(1001) + q^{3n-\frac{11}{2}}(\bar{1}11\bar{3}13\bar{3}02\bar{2}01\bar{1}) + q^{4n-\frac{9}{2}}(\bar{3}16\bar{8}\bar{5}12\bar{4}\bar{9}9\bar{1}\bar{6}50\bar{2}1) + q^{5n-\frac{11}{2}}(10\bar{5}412\overline{13}\overline{10}23\bar{3}\overline{18}142\overline{11}61\bar{3}1)$ $+q^{6n-\frac{7}{2}}(1\bar{3}\bar{1}11\bar{6}\overline{16}197\overline{22}109\overline{12}43\bar{3}1) + q^{7n-\frac{1}{2}}(\bar{1}21\bar{6}27\bar{7}\bar{2}6\bar{3}\bar{1}2\bar{1})$	
10_{59}	5	$(1\bar{1}) + q^{n}(5\bar{3}1) + q^{2n}(1\bar{1})$	Same as 10_{106}, $n = 2$
		$q^{-n-\frac{11}{2}}(\bar{1}11\bar{2}01\bar{1}) + q^{-\frac{13}{2}}(11\bar{5}19\bar{6}\bar{5}8\bar{1}\bar{3}2) + q^{n-\frac{13}{2}}(\bar{2}35\overline{14}\bar{4}23\bar{9}\overline{18}152\overline{10}31\bar{1}) + q^{2n-\frac{13}{2}}(1\bar{3}113\overline{14}\overline{18}374\overline{37}2014\overline{17}54\bar{3}1)$ $+q^{3n-\frac{9}{2}}(\bar{1}06\bar{6}\overline{15}1812\overline{29}216\overline{11}\bar{2}4\bar{2}) + q^{4n-\frac{3}{2}}(2\bar{2}\bar{4}74\bar{8}15\bar{2}01) + q^{5n+\frac{3}{2}}(\bar{1}11\bar{2}01\bar{1})$	
10_{60}	5	$(\bar{5}3\bar{1}) + q^{n}(\bar{3}2) + q^{2n}(\bar{1})$	Same as 10_{83}, $n = 2$
		$q^{n-\frac{7}{2}}(\bar{1}11\bar{2}11\bar{1}) + q^{2n-\frac{7}{2}}(1\bar{2}\bar{1}6\bar{2}\bar{4}50\bar{2}1) + q^{3n-\frac{3}{2}}(\bar{1}\bar{1}40\bar{5}22\bar{2}) + q^{4n+\frac{3}{2}}(20\bar{2}11) + q^{5n+\frac{9}{2}}(\bar{1})$	Same as 7_6
10_{71}	5	$q^{-n}(\bar{1}1) + (\bar{5}3\bar{1}) + q^{n}(\bar{1}1)$	Same as 10_{104}, $n = 2$ falsely amphichiral
		$q^{-n-\frac{3}{2}}(\bar{1}) + q^{-\frac{7}{2}}(110\bar{1}01) + q^{n-\frac{7}{2}}(\bar{1}\bar{1}3\bar{2}\bar{4}21\bar{2}) + q^{2n-\frac{5}{2}}(1\bar{1}13\bar{3}04\bar{1}\bar{1}1) + q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	Same as 9_{44}
10_{73}	5	$q^{n}(5\bar{3}1) + q^{2n}(3\bar{2}) + q^{3n}(1)$	Same as 10_{86}, $n = 2$
		$q^{n-\frac{13}{2}}(\bar{1}30\bar{9}88\overline{15}310\bar{7}\bar{1}3\bar{1}) + q^{2n-\frac{13}{2}}(1\bar{3}38\overline{21}237\overline{28}\overline{21}37\bar{6}\overline{17}111\bar{3}1) + q^{3n-\frac{9}{2}}(\bar{2}27\overline{19}\bar{4}40\overline{19}\overline{37}348\overline{23}55\bar{3}) + q^{4n-\frac{5}{2}}(14\bar{9}\bar{3}27\bar{7}\overline{26}2011\overline{12}13)$ $+q^{5n+\frac{1}{2}}(\bar{3}\bar{2}9\bar{3}\overline{13}55\bar{5}\bar{1}) + q^{6n+\frac{7}{2}}(30\bar{3}22) + q^{7n+\frac{13}{2}}(\bar{1})$	

Table 3. (Continued)

Knot	BI	$\{1\}su(n)_q$ polynomial $\{2\}su(n)_q$ polynomial	Comments Comments
10_{81}	5	$q^{-n}(\bar{1}1)+(\bar{5}4\bar{1})+q^{n}(\bar{1}1)$ $q^{-3n-\frac{13}{2}}(\bar{1}11\bar{2}01\bar{1})+q^{-2n-\frac{15}{2}}(1\bar{1}\bar{5}56\overline{10}\bar{1}8\bar{3}\bar{2}2)+q^{-n-\frac{15}{2}}(\bar{2}62\overline{20}1425\overline{33}\bar{6}29\overline{10}\bar{9}70\bar{1})+q^{-\frac{15}{2}}(1\bar{4}313\overline{26}\bar{2}51\overline{34}\,\overline{34}51\bar{2}\overline{26}133\bar{4}1)$ $+q^{n-\frac{11}{2}}(\bar{1}07\bar{9}\overline{10}29\bar{6}\overline{33}2514\overline{20}26\bar{2})+q^{2n-\frac{5}{2}}(2\bar{2}\bar{3}8\bar{1}\overline{10}65\bar{5}\bar{1}1)+q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	Same as 10_{109}, $n=2$
10_{83}	4	$(\bar{4}3\bar{1})+q^{n}(\bar{3}2\bar{1})$ $q^{-n-\frac{15}{2}}(\bar{1}3\bar{1}\bar{8}114\overline{17}79\bar{9}03\bar{1})+q^{-\frac{13}{2}}(\bar{2}41\overline{16}1617\overline{36}331\overline{17}\,\overline{10}12\bar{1}\bar{3}1)+q^{n-\frac{15}{2}}(1\bar{1}\bar{4}82\overline{23}1724\overline{42}\bar{2}37\overline{16}\,\overline{14}120\bar{3}1)$ $+q^{2n-\frac{11}{2}}(1\bar{3}19\overline{14}\bar{2}28\overline{21}\,\overline{17}31\bar{4}\overline{16}92\bar{3}1)+q^{3n-\frac{5}{2}}(\bar{1}20\bar{5}62\overline{10}55\bar{6}02\bar{1})$	Same as 10_{60}, $n=2$
10_{91}	3	$(\bar{4}4\bar{2}1)$ $q^{-n-\frac{21}{2}}(\bar{1}2\bar{1}\bar{5}81\overline{16}1111\overline{21}413\overline{13}16\bar{5}02\bar{1})+q^{-\frac{21}{2}}(1\bar{2}16\bar{9}119\overline{20}\bar{7}39\overline{21}\,\overline{22}38\bar{7}\overline{19}18\bar{1}\bar{8}60\bar{2}1)+q^{n-\frac{15}{2}}(\bar{1}2\bar{1}\bar{5}70\overline{14}135\overline{21}101\overline{15}27\bar{5}02\bar{1})$	Same as 10_{43}, $n=2$ falsely amphichiral
10_{94}	3	$q^{n}(4\bar{4}2\bar{1})$ $q^{-3n-\frac{23}{2}}(\bar{1}2\bar{1}\bar{4}7\bar{3}\overline{10}16\bar{3}\overline{18}165\overline{16}56\bar{6}02\bar{1})+q^{-2n-\frac{23}{2}}(1\bar{2}15\bar{9}313\overline{21}822\overline{32}432\overline{23}\,\overline{10}23\bar{5}\bar{8}70\bar{2}1)+q^{-n-\frac{17}{2}}(\bar{1}2\bar{1}\bar{4}7\bar{3}\bar{8}12\bar{5}\overline{12}140\overline{15}75-702\bar{1})$	Same as 10_{41}, $n=2$
10_{100}	3	$q^{n}(1)+q^{2n}(\bar{3}4\bar{2}1)$ $q^{n+\frac{1}{2}}(1)+q^{2n-\frac{1}{2}}(1001)+q^{3n-\frac{17}{2}}(\bar{1}22\bar{6}19\bar{7}\bar{4}11\bar{3}\bar{6}8\bar{2}\bar{4}5\bar{2}\bar{2}2\bar{1})+q^{4n-\frac{17}{2}}(1\bar{2}\bar{1}8\bar{5}\overline{12}185\overline{24}1216\overline{20}514\overline{14}68\bar{8}42\bar{2}1)$ $+q^{5n-\frac{11}{2}}(\bar{1}21\bar{7}310\overline{12}\bar{6}15\bar{5}\overline{12}10\bar{3}\bar{6}6\bar{3}\bar{2}2\bar{1})$	Different to ADW for $n=2$
10_{103}	4	$q^{n}(4\bar{2}1)+q^{2n}(3\bar{2}1)$ $q^{n-\frac{13}{2}}(\bar{1}21\bar{6}36\bar{8}16\bar{4}\bar{1}2\bar{1})+q^{2n-\frac{11}{2}}(\bar{1}43\overline{13}321\overline{15}\,\overline{10}20\bar{4}\bar{8}70\bar{2}1)+q^{3n-\frac{13}{2}}(1\bar{1}\bar{4}86\overline{22}332\overline{21}\,\overline{19}27\bar{3}\overline{11}80\bar{2}1)$ $+q^{4n-\frac{9}{2}}(1\bar{3}010\overline{11}\,\overline{12}27\bar{4}\overline{27}206\overline{15}62\bar{3}1)+q^{5n-\frac{3}{2}}(\bar{1}21\bar{6}37\bar{9}\bar{1}7\bar{4}\bar{1}2\bar{1})$	Same as 10_{40}, all n
10_{104}	3	$(\bar{5}4\bar{2}1)$ $q^{-n-\frac{21}{2}}(\bar{1}2\bar{1}\bar{5}91\overline{18}1413\overline{26}616\overline{15}17\bar{5}02\bar{1})+q^{-\frac{21}{2}}(1\bar{2}16\overline{10}121\overline{24}\bar{9}46\overline{27}\,\overline{28}45\bar{9}\overline{23}20\bar{1}\bar{9}60\bar{2}1)+q^{n-\frac{15}{2}}(\bar{1}2\bar{1}\bar{5}80\overline{16}167\overline{26}1214\overline{17}28\bar{5}02\bar{1})$	Same as 10_{71}, $n=2$ falsely amphichiral

Table 3. (Continued)

Knot	BI	$\{1\}su(n)_q$ polynomial $\{2\}su(n)_q$ polynomial	Comments Comments
10_{106}	3	$q^n(5\bar{4}2\bar{1})$	Same as 10_{59}, $n=2$
		$q^{-3n-\frac{23}{2}}(\bar{1}2\bar{1}\bar{4}8\bar{4}\overline{11}20\bar{4}\overline{22}206\overline{18}67\bar{6}02\bar{1})+q^{-2n-\frac{23}{2}}(1\bar{2}15\overline{10}414\overline{26}1027\overline{40}438\overline{28}\overline{12}25\bar{6}\bar{9}70\bar{2}1)+q^{-n-\frac{17}{2}}(\bar{1}2\bar{1}\bar{4}8\bar{4}\bar{9}16\bar{5}\overline{15}181\overline{17}86\bar{7}02\bar{1})$	
10_{109}	3	$(\bar{5}5\bar{2}1)$	Same as 10_{81}, $n=2$
		$q^{-n-\frac{21}{2}}(\bar{1}2\bar{1}\bar{6}101\overline{23}1815\overline{34}820\overline{21}09\bar{6}\bar{1}2\bar{1})+q^{-\frac{21}{2}}(1\bar{2}17\overline{11}127\overline{29}\overline{10}60\overline{33}\overline{33}60\overline{10}\overline{29}271\overline{11}71\bar{2}1)+q^{n-\frac{15}{2}}(\bar{1}2\bar{1}\bar{6}90\overline{21}208\overline{34}1518\overline{23}110\bar{6}\bar{1}2\bar{1})$	
10_{112}	3	$q^n(6\bar{4}3\bar{1})$	
		$q^{n-\frac{19}{2}}(\bar{1}30\overline{10}911\overline{22}124\overline{18}\bar{7}20\overline{11}\bar{5}11\bar{6}\bar{1}3\bar{1})+q^{2n-\frac{19}{2}}(1\bar{3}010\overline{13}\bar{8}34\overline{15}\overline{36}487\overline{49}3414\overline{32}195\overline{14}71\bar{3}1)$ $+q^{3n-\frac{13}{2}}(\bar{1}30\bar{9}108\overline{24}625\overline{27}\bar{6}24\overline{14}\bar{5}11\bar{6}\bar{1}3\bar{1})$	Different to ADW for $n=2$
10_{125}	3	(101)	Falsely amphichiral
		$q^{-n-\frac{17}{2}}(\bar{1}00\bar{1}\bar{1}0\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}0\bar{1})+q^{-\frac{17}{2}}(1002212332222101)+q^{n-\frac{11}{2}}(\bar{1}00\bar{1}\bar{2}\bar{1}0\bar{1}\bar{2}000\bar{1})$	
10_{129}	4	$(\bar{1}1)+q^n(\bar{1}1)$	Same as 8_8, all n
		$q^{-n-\frac{15}{2}}(1\bar{3}25\bar{8}09\bar{7}\bar{2}5\bar{2}\bar{1}1)+q^{-\frac{15}{2}}(\bar{1}4\bar{4}\bar{7}18\bar{4}\overline{22}235\overline{22}126\bar{9}22\bar{1})+q^{n-\frac{11}{2}}(10\bar{6}89\overline{20}420\overline{15}\bar{2}11\bar{4}\bar{1}2)+q^{2n-\frac{1}{2}}(2\bar{2}\bar{3}40\bar{3}10\bar{1})+q^{3n+\frac{1}{2}}(\bar{1}10\bar{2}11\bar{1})$	
10_{132}	4	$q^n(1)+q^{2n}(01)$	Same as 5_1, all n
		$q^{n+\frac{1}{2}}(1)+q^{2n-\frac{5}{2}}(10\bar{1}11)+q^{3n-\frac{3}{2}}(11\bar{1}02)+q^{4n-\frac{1}{2}}(1000011001)+q^{5n+\frac{5}{2}}(\bar{1}0\bar{1}\bar{1}00\bar{1})$	
10_{137}	5	$(\bar{1})+q^n(\bar{2}1)+q^{2n}(\bar{1})$	Same as 10_{155}, $n=2$
		$q^{-n-\frac{3}{2}}(\bar{1})+q^{-\frac{7}{2}}(10\bar{1}\bar{1}01)+q^{n-\frac{7}{2}}(\bar{1}13\bar{5}\bar{4}41\bar{2})+q^{2n-\frac{1}{2}}(3\bar{1}\bar{6}35\bar{2}\bar{1}1)+q^{3n-\frac{1}{2}}(\bar{1}12\bar{5}\bar{2}40\bar{2})+q^{4n+\frac{5}{2}}(2\bar{1}\bar{2}21)+q^{5n+\frac{11}{2}}(\bar{1})$	
10_{155}	3	$q^n(\bar{2}1\bar{1})$	Same as 10_{137}, $n=2$
		$q^{n-\frac{11}{2}}(\bar{1}0\bar{2}02\bar{5}\bar{2}4\bar{2}\bar{3}11\bar{1})+q^{2n-\frac{11}{2}}(1\bar{1}12\bar{2}41\bar{5}54\bar{5}04\bar{1}\bar{1}1)+q^{3n-\frac{5}{2}}(\bar{1}1\bar{1}1\bar{2}\bar{2}23\bar{1}\bar{3}11\bar{1})$	
10_{156}	4	$q^n(3\bar{2}1)$	Same as 8_{16}, all n
		$q^{n-\frac{13}{2}}(\bar{1}21\bar{5}25\bar{5}04\bar{3}02\bar{1})+q^{2n-\frac{13}{2}}(1\bar{2}07\bar{6}\bar{8}150\overline{14}102\bar{7}50\bar{2}1)+q^{3n-\frac{7}{2}}(\bar{1}21\bar{7}39\overline{10}\bar{2}8\bar{4}\bar{1}2\bar{1})+q^{4n+\frac{3}{2}}(\bar{1}11\bar{2}01)$	

The polynomial of a mirror image to a knot is obtained by substituting q^{-1} for q in the knot polynomial. Knots which are amphichiral have polynomials symmetric in q and q^{-1}. The $\{1\}su(n)_q$ polynomials for the knots 9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}, and 10_{125} are symmetric in q and q^{-1}, as shown in table 3, but these knots are not amphichiral. The $\{2\}su(n)_q$ polynomial does correctly reflect the amphichirality/non-amphichirality of all the knots having ten or fewer crossings.

Table 2 summarizes the differences between the $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials. Table 3 gives the $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials for all knots mentioned in table 2 and also for 10_{100} and 10_{112}. Full tables of $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials for all knots of ten or fewer crossings are available from the authors (E-mail address: P.Butler@Phys.Canterbury.AC.NZ).

7. Conclusions

Knot polynomials may be obtained from the R-matrices of q-deformed algebras. The one-variable polynomials based on the $\{1\}$ representation of $su(n)_q$ are a special case of the two-variable HOMFLY polynomial. For knots of ten or fewer crossings and with $m > 2$ the only pairs of knots with the same $\{1\}su(n)_q$ polynomial are those pairs with the same HOMFLY polynomial. Other pairs for the Jones polynomial, which is equivalent to the $\{1\}su(2)_q$ polynomial, are distinguished for higher values of n. These one-variable polynomials are as effective at distinguishing knots of ten or fewer crossings as the two-variable HOMFLY polynomial.

The $N = 3$ $\{2\}su(n)_q$ polynomials are calculated by recursion for all knots of ten or fewer crossings. The recursion has been automated, with the algebraic package MAPLE being used. We use a similar method to that of Guadagnini [19] to find two skein-type relations, which together with the Alexander–Conway skein relation are sufficient to determine polynomials for all knots with ten or fewer crossings. However, the properties of R-matrices and coupling coefficients for q-deformed algebras are used rather than appealing to conformal field theory.

Akutsu and Wadati [23] show that the $\{2\}su(2)_q$ polynomial distinguishes a pair of knots with braid index 3 having the same Jones and HOMFLY polynomials. The $\{2\}su(2)_q$ was thought to be more powerful than any of the $\{1\}su(n)_q$ polynomials. Extending the calculation both to all the knots of ten or fewer crossings and to all $\{2\}su(n)_q$ polynomials shows that all pairs for the HOMFLY polynomial of knots of ten or fewer crossings are distinguished by the $\{2\}su(n)_q$ polynomial. However, four new pairs for the $\{2\}su(n)_q$ polynomial have been found. The $\{2\}su(n)_q$ polynomials do correctly predict non-amphichirality for all knots of ten or fewer crossings.

From this data set, it seems that the Jones polynomial or $\{1\}su(2)_q$ polynomial is anomalous in the number of pairs obtained. For other values of n or other representations there are a similar number of pairs, far fewer than for the Jones polynomial. It seems unlikely that the $\{3\}su(n)_q$ polynomial or higher representation polynomials would have fewer pairs. The $\{1\}$ and $\{2\}$ polynomials together are sufficient to distinguish all knots of ten or fewer crossings.

References

[1] Alexander J W 1928 *Trans. Am. Soc.* **30** 275
[2] Jones V F R 1985 *Bull. Am. Math. Soc.* **12** 103
[3] Freyd P, Yetter D, Hoste J, Lickorish W B R, Millet K and Ocneanu O 1985 *Bull. Am. Math. Soc.* **12** 183

[4] Kauffman L H 1987 *Topology* **26** 395
[5] Sklyanin E K 1982 *Funkt. Anal. Pril.* **16** 27 (Engl. transl. 1982 *Funct. Anal. Appl.* **16** 263)
[6] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[7] Drinfel'd V G 1985 *Sov Math.–Dokl.* **32** 254
[8] Rosso M 1989 *Commun. Math. Phys.* **124** 307
[9] Lienert C R and Butler P H 1992 *J. Phys. A: Math. Gen.* **25** 1223
[10] Lienert C R and Butler P H 1992 *J. Phys. A: Math. Gen.* **25** 5577
[11] Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 839
[12] Turaev V G 1988 *Invent. Math.* **92** 572
[13] Reshetikhin N Y 1987 lOMI St. Petersburg E-4-87 and E-17-87 (unpublished)
[14] Broda B 1991 *Phys. Lett.* **270B** 116
[15] Artin E 1925 *Abh. Math. Sem. Univ. Hamburg* **4** 47
[16] Markov A A 1945 *Trudy Math. Inst. Steklov* **16** 1
[17] Hou B Y and Ma Z Q 1990 *Commun Theor. Phys.* **13** 181
[18] Zhang R B, Gould M D and Bracken A J 1991 *Commun. Math. Phys.* **137** 13
[19] Guadagnini E 1992 *Int. J Mod. Phys.* A **7** 877
[20] Nomura M 1989 *J. Math. Phys.* **30** 2397
[21] Jones V F R 1987 *Ann Math.* **126** 335
[22] MAPLE V, Waterloo Maple Software, University of Waterloo, ©1981–90
[23] Akutsu Y, Deguchi T and Wadati M 1987 *J. Phys. Soc. Japan* **56** 3464